Problem 1.3 Show that the spectral density, u_w (equation 1.2.4) peaks at E_{ph} = 2.82 kT. Note that a numeric iteration is required.

Solution

The spectral density, u_w, can be rewritten as a function of $x = \frac{\hbar w}{kT}$

$$u_w = \frac{k^3 T^3}{\hbar^2 p^2 c^3} \frac{x^3}{\exp(x) - 1}$$

The maximum of this function is obtained if its derivative is zero or:

$$\frac{du_w}{dx} = \frac{3x^2}{\exp(x) - 1} - \frac{x^3 \exp(x)}{(\exp(x) - 1)^2} = 0$$

Therefore x must satisfy:

$$3 - 3\exp(-x) = x$$

This transcendental equation can be solved starting with an arbitrary positive value of x. A repeated calculation of the left hand side using this value and the resulting new value for x quickly converges to $x_{\max}$ = 2.82144. The maximum spectral density therefore occurs at:

$$E_{ph,\max} = x_{\max} kT = 2.82144 \text{ kT}$$